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APPENDIX B

Basic Theory of Stirling-Cycle Refrigerator

In this section a brief outline of the basic principles of the Stirling cycle is presented.[1,2] From an elementary theory the general properties of the cycle will be derived with a discussion of the most important losses. It should be noted that the presentation is essentially taken from Part Two of reference 1. For a more extensive treatment of the subject reference should be made to the original paper[3].

Fundamental Cycle

In the ideal Stirling cycle, the cold is produced by the reversible expansion of a gas. The gas performs a closed cycle, during which it is alternately compressed at ambient temperature in a compression space and expanded at the desired low temperature in an expansion space, thereby reciprocating between these spaces through one connecting duct, wherein a regenerator provides for the heat exchange between the outgoing and the returning gas flow. Figure 1 shows stages in carrying out the ideal cycle.

In this diagram, A is a cylinder, closed by the piston B, and containing a nearly perfect gas. A second piston, the displacer C, divides the cylinder into two spaces, D at room temperature and E at the low temperature, connected by the annular passage F. This passage contains the regenerator G, a porous mass with a high heat capacity; the temperature in the passage is shown in the graph. The cycle, consisting of four phases, runs as follows:

I Compression in space D by the piston B; the heat of compression is discharged through the cooler H.

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1 Dr. J. W. L. Köhler "The Gas Refrigerating Machine and Its Position in Cryogenic Technique," Progress in Cryogenics 2, 41-67 (1960) London, Heywood and Company Ltd.

2 A. Daniels, "Cryogenics for Electro-Optical Systems," Electro-Optical Systems Design, vol. 3, pp. 12-20, July 1971.

3 J. W. L. Köhler and C. O. Jonkers, Philips Tech Rev. 16 69-78, 105-115 (1954)

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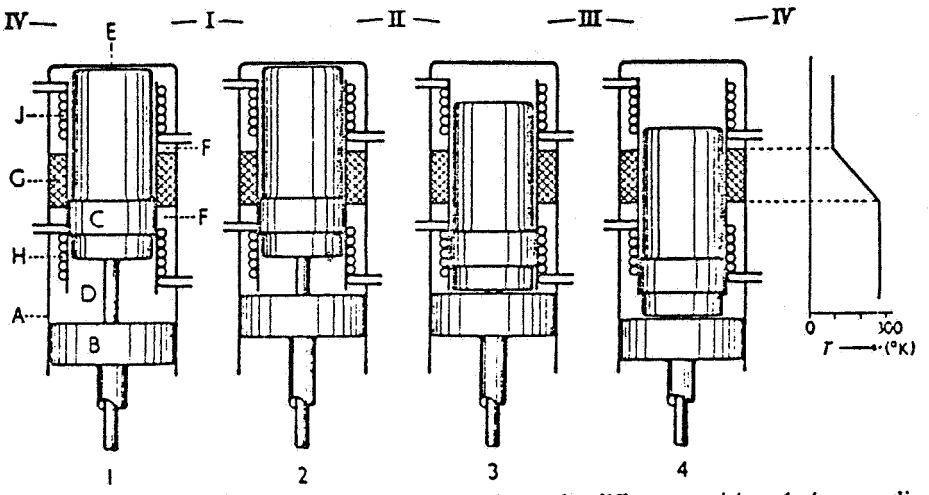

Figure 1. *The ideal Stirling cycle. The Figure shows the different positions 1–4, according to Figure 2. I–IV refer to the four phases of the cycle. The graph on the right shows the temperature distribution in the machine*

II Transfer of the gas through the regenerator to space E by movement of the displacer. The gas is reversibly cooled down in the regenerator, the heat of the gas being stored in the regenerator mass.

III Expansion in the cold space by the combined movement of the piston and the displacer; the cold produced is discharged through the freezer J, and utilized.

IV Return of the gas to space D; thereby the gas is reheated, the heat stored in the regenerator being restored to the gas.

With no regenerator present the gas flowing to the expansion space would arrive there at ambient temperature, whereas the returning gas would arrive in the compression space at the low temperature; this effect would cause such a tremendous cold loss that the whole process would become futile. In an ideal regenerator, a temperature gradient is established in the direction of the gas flow. This causes the gas to be cooled down reversibly, so that it arrives in the expansion space with the temperature prevailing there.

Figure 2 shows the p-V diagram of this schematic cycle, neglecting the dead space. It consists of two isotherms and two isochores; at T_C (compression temperature) the amount of heat Q_C is rejected, at T_E (expansion temperature) the amount of heat Q_E is absorbed.

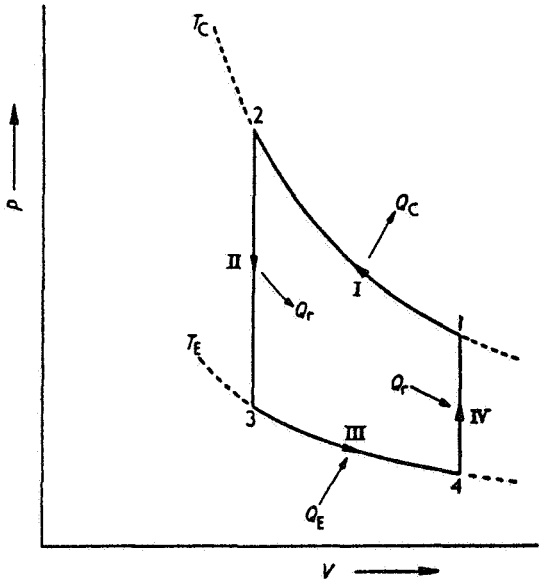

Figure 2. p-V *diagram of the ideal cycle between the temperatures* T_C *and* T_E, *where the amounts of heat* Q_C *and* Q_E *are discharged and absorbed, respectively. The heat* Q_r, *rejected in phase II, is stored in the regenerator and absorbed in phase IV*

Actually, the discontinuous movement of the pistons is difficult to achieve. In practice, the pistons are actuated by a crank mechanism and are thus moving harmonically; for the machine to act as a refrigerator, the expansion space E (Figure 1) has to lead in phase with respect to the compression space D. The harmonic motion and the volume of the heat exchangers (the so-called 'dead space') cause the four phases of the cycle to merge somewhat, so that they cannot be distinguished very well; the gas is not exclusively compressed in the compression space, but also in the expansion space, and the same holds for the expansion. It can be shown however that the difference in output between the discontinuous and the harmonic process is only a few percent.

The fundamental cycle is explained here with the help of the displacer machine. The reason is that actual machines are also of this type. It will be obvious, however, that the cycle may be described more generally by two synchronously changing volumes interconnected by a cooler, a regenerator, and a freezer, whereby the volume that is leading in phase becomes the expansion space wherein the cold is produced. Formulae for the pressure variation, the refrigerating capacity, and the shaft power for this general case will be given in the next section.

Performance of the Ideal (Isothermal) Cycle

In the ideal machine, the thermal contact in the heat exchangers is assumed to be perfect, so that the gas temperature there is equal to the temperature of the walls; the same temperature is supposed to prevail in the adjoining cylinders, which temperatures thus are constant with time. The regenerator is also assumed to be perfect, so that no regeneration loss occurs and the gas temperature there is also constant with time. The working fluid is supposed to be a nearly perfect gas; this condition can be met sufficiently by using hydrogen or helium in practice.

(1) Pressure variation. For the expansion space V_E (temperature T_E) and the compression space V_C (temperature T_C) we write

$$\left.\begin{aligned} V_E &= \tfrac{1}{2} V_0(1 + \cos\alpha) \\ V_C &= \tfrac{1}{2} wV_0[1 + \cos(\alpha-\phi)] \end{aligned}\right\} \qquad (1)$$

where V_0 is the maximum volume of the expansion space, w the ratio of the swept volumes of the compression and the expansion space, and ϕ the phase difference between these spaces; the crank angle α (= 0 for $V_E = V_{max}$) changes linearly with time ($\alpha = \omega t$). The volumes of the freezer, the regenerator, and the cooler, which form the connecting channel (the dead space), will be indicated by V_S with the corresponding temperature T_S. The variation of the pressure p with time (or α) now follows from the condition that the mass of the system as a whole is constant:

$$\frac{M}{R}p\left[\frac{V_E}{T_E}+\frac{V_C}{T_C}+\sum\frac{V_s}{T_s}\right] = \text{constant} = C.\frac{M}{R}\frac{V_0}{2T_C}$$

where M is the molecular weight of the gas, R is the gas constant, and C is a constant. After introduction of V_E and V_C from equation (1) and of the symbols

$$\tau = \frac{T_C}{T_E}, \quad \text{the temperature ratio}$$

and

$$s = \sum\frac{V_s}{V_0}.\frac{T_C}{T_s}, \quad \text{the total dead space}$$

(reduced to the swept volume of the expansion space V_0 and normalized to the temperature of the compression space T_C), this reduces to

$$\frac{C}{p} = \tau\cos\alpha+w\cos(\alpha-\phi)+\tau+w+2s$$

This expression is easily transformed into

$$\frac{C}{p} = A\cos(\alpha-\theta)+B = B[1+\delta\cos(\alpha-\theta)]$$

with the abbreviations

$$A = \sqrt{(\tau^2+w^2+2\tau w\cos\phi)}; \qquad B = \tau+w+2s$$

$$\frac{A}{B} = \delta; \qquad \tan\theta = \frac{w\sin\phi}{\tau+w\cos\phi}$$

The constant A may be interpreted as two times the total change of volume, while B equals two times the mean total volume, both reduced to V_o and normalized to T_C. For the pressure p we thus find

$$p = \frac{C}{B}\frac{1}{1+\delta\cos(\alpha-\theta)}$$

which may be written more conveniently

$$p = \frac{p_{max}(1-\delta)}{1+\delta\cos(\alpha-\theta)} = \frac{p_{min}(1+\delta)}{1+\delta\cos(\alpha-\theta)} \qquad (2)$$

with the introduction of p_{max} and p_{min} for the maximum and minimum values of the pressure during the cycle.

Expression (2) shows that the pressure variation with time is not purely harmonic. In practice, the deviation of harmonic behaviour is rather small however (the function is symmetrical with respect to p_{max} and p_{min}, which points are 180° apart), as the value of δ seldom exceeds 0.4. This means that the pressure ratio of this type of machine

$$\frac{p_{max}}{p_{min}} = \frac{1+\delta}{1-\delta}$$

is about 2, a remarkably low value compared with what is normal in refrigerating apparatus. The presence of the phase angle θ shows that the pressure variation is not in phase with the variation of the expansion or the compression space; it is easily checked that its phase is intermediate between that of these spaces ($\theta \to \phi$ for $\tau \to 0$). It will be found, for this reason, heat is absorbed in the expansion space and liberated in the compression space. For later reference, given here is the expression for the mean pressure p_m, deduced by integrating the pressure with respect to the crank angle α:

$$p_m = p_{max}\sqrt{\left(\frac{1-\delta}{1+\delta}\right)} \qquad (3)$$

(2) Heat absorption in cylinders. The heat absorbed per cycle in the expansion space (Q_E) and in the compression space (Q_C, a negative quantity) is given by

$$Q_E = \oint p\,dV_E, \quad Q_C = \oint p\,dV_C$$

That these expressions, which are normally used for spaces with a constant gas content, may also be used in the case where gas enters and leaves the space can be proved by an involved thermodynamic reasoning which is omitted here. It is easily seen that the value of the integrals depends only on the components of p which have the same phase as dV_E and dV_C; this means that $0 < \theta < \phi$ if the machine has to operate as a refrigerator. Evaluation of the integrals leads to

$$\left.\begin{aligned} Q_E &= \frac{\pi}{a} V_0 p_m \frac{w \sin\phi}{B} \\ Q_C &= -\tau Q_E \end{aligned}\right\} \qquad \ldots(4)$$

with $a = 1 + \sqrt{(1-\delta^2)} \simeq 2$

For practical use, equation (4) may be transformed into q_E, the heat absorbed per second (or the refrigerating capacity) by inserting n, the number of revolutions per minute. If V_0 is expressed in cm^3, p_m in kg cm^{-2}, and q_E in watts, it is found that

$$q_E = \frac{5 \cdot 136}{a} V_0 p_m \frac{n}{1000} \frac{w \sin\phi}{B} \quad \text{(in watts)} \qquad \ldots(4a)$$

Expression (4a) shows that, besides with V_0 and n, the output is proportional to p_m and inversely proportional to the function B, which was defined as the mean reduced and normalized volume. The consequences of the dependence on the mean pressure and the temperature ratio (τ is also contained in s), which constitutes one of the features of the process, will be discussed at more length later. According to expression (4a) an increase of the dead space s reduces the output, as could be expected. A discussion on the influence of the design constants w and ϕ would be very elaborate and lengthy and is therefore outside the scope of this report; their choise depends largely on the losses of the process.

(3) Shaft power and efficiency. For the work W needed to drive the machine one may write

$$W = -Q_E - Q_C$$

Thus

$$W = (\tau - 1)Q_E = \frac{\pi}{a}(\tau - 1)V_0 p_m \frac{w \sin\phi}{B} \qquad \ldots(5)$$

Using equation (5) the efficiency of the cycle is

$$\eta = \frac{Q_E}{W} = \frac{1}{\tau - 1} = \frac{T_E}{T_C - T_E} \qquad \ldots(6)$$

The efficiency of the ideal cycle thus equals that of the Carnot cycle; this is obvious as the cycle is completely reversible.

Although in actual machines the performance is reduced by losses, to be discussed in the next section, the ideal cycle has to be considered as the reference process, because most essential facts can be deduced from it.

Before closing this section two remarks on the representation of the process will be made. The first remark concerns the representation in a thermodynamic diagram (e.g., p vs V, T vs S, etc.). These diagrams always relate to a fixed quantity of the working fluid, which has to be in internal equilibrium (equal p and T throughout); this quantity is passed through a number of thermodynamic states and (at least in a closed system) is returned ultimately to its initial state, so that a cycle is described. Looking at the Stirling process, it is found that, in so far as cyclic behaviour is concerned, nothing abnormal is at hand. However, the system is not homogeneous at all, as different parts of it have a different temperature. As a consequence, different gas particles describe completely different cycles between different temperatures; to mention only two extreme examples, some particles are reciprocating between the compression space and a point in the cooler, while other particles are reciprocating between a point in the freezer and the expansion space. That, because of this situation, normal diagrams have lost their value is shown by the fact that one would have to draw an infinite number of diagrams for the diverse particles with different cycles, which obviously leads

nowhere. The only diagram which still has some sense is the p-V diagram, as the system is homogeneous in p. But in such a diagram one is not allowed to draw isotherms or adiabates, as these lines have lost their meaning. To avoid this difficulty when drawing Figure 2, the dead space was assumed to be zero; in that case isotherms may be drawn, since only then the gas is at thermal equilibrium during the compression and the expansion. When the adiabatic losses are discussed in the next section, this subject will have to be returned to. The second remark concerns the description of the cycle in a schematic form. The transfer of the gas from the compression space to the expansion space and vice versa is performed with constant volume of the gas; this is the simplest representation, as the transfer can be effected by the movement of the displacer only. But this way of transfer is only one example of a multitude of possibilities which exhibit the common property that regeneration is possible. As another example consider transfer at constant pressure, which way of transfer approximates the harmonic cycle much better. The only reason why this manner of transfer is not used to explain the cycle is that it can only be accomplished by simultaneous movement of the piston and the displacer, which obviously is more complicated. This point is stressed because at many places in literature the transfer with constant volume is considered to be one of the main characteristics of the Stirling cycle, to contrast it with the cycle of Claude (with separate compressor and expander), where the transfer is performed at constant pressure. As explained, this view does not correctly locate the distinction between the two cycles, the real difference being that in the Stirling cycle the gas is reciprocating through one connecting duct, wherein the heat exchange is brought about by regeneration, whereas in the Claude cycle the connecting circuit consists of a counter-flow heat exchanger (with two channels). Apart from the restriction contained in the first remark the Stirling cycle thus closely resembles the Claude cycle thermodynamically.

The Actual Cycle

As stated already, the actual cycle differs from the ideal one by the occurrence of losses. For a fuller discussion on this subject reference is made to the original paper[2], here only the most characteristic effects will be treated.

The losses can affect the process in two different ways, viz., by increasing the shaft power and by decreasing the refrigerating capacity; the smaller the ideal values of these quantities, the more pronounced will be the relative effect. Figure 3 illustrates that increase of the shaft power exerts the greatest influence at high refrigerating temperatures, while decrease of the output creates the most adverse effects at low temperatures. In this way a temperature range arises, the "optimum working range", wherein the actual efficiency differs least from the ideal one (i.e., where the figure of merit is highest). This range may be shifted both to higher and to lower temperatures by suitable design; moreover, its limits greatly depend on present and future technological possibilities.

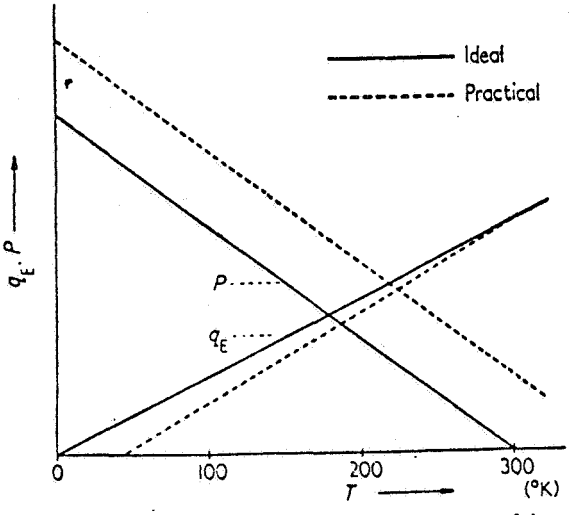

Figure 3. *Graphs of the shaft power* P *and the refrigerating capacity* q_E *as a function of temperature. The full lines apply to the ideal process, the dotted lines to an actual machine with losses*

The increase in shaft power is mainly due to three causes, namely, the mechanical loss of the drive, the flow loss (that is, the power needed to force the gas through the narrow connecting circuit) and the adiabatic loss. The first two losses need no further comment, but the adiabatic loss will be discussed at greater length. In the ideal isothermal process it was assumed that the temperature in the cylinders is constant with time. This means that the thermal contact between the gas and the wall in the cylinder spaces is assumed to be so perfect that these walls can be employed as well to establish the contact between the gas and the surroundings, that is, to serve as cooler and freezer. In this case, separate heat exchangers are of no use and must be omitted; this ideal machine thus consists of the two cylinders and the regenerator and is therefore referred to as the "three-element machine".

Actually, the separate heat exchangers are introduced because the thermal contact between the gas and the cylinder walls is always so poor that insufficient heat exchange with the surroundings can be established through these walls; for this case the name "five-element machine" is used. As a consequence, the temperature of the gas in the cylinders changes nearly adiabatically with time. This adiabatic behavior has only a minor influence on the efficiency (it would be too involved to give the full explanation here; it is based on the fact that these processes occur in both cylinders with the same phase, so that the temperature ratio is independent of time) but in this case the heat (or the cold) must be transported from the cylinders to the heat exchangers. This transport can only be effected by the reciprocating gas which performs it by assuming different temperatures when flowing in opposite directions, with the result that the mean temperature in each cylinder deviates from that in the adjoining heat exchanger. Figure 4 shows schematically the temperature distribution in the machine. It will be observed that the mean temperature in the expansion cylinder is lower than that of the freezer and that the mean temperature in the compression cylinder is higher than that of the cooler. This means that the ratio of the cylinder temperatures is higher than that of the temperatures of the

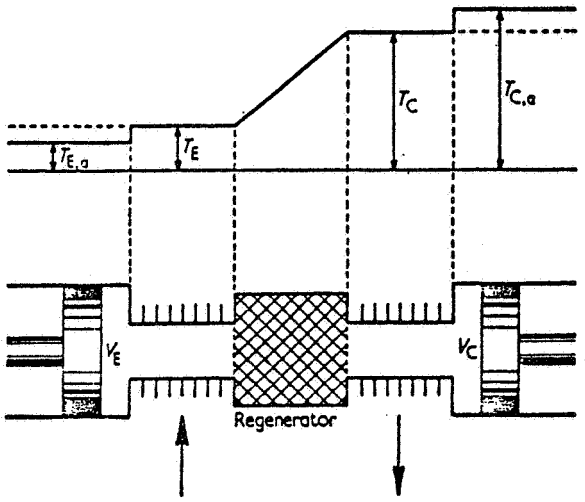

Figure 4. *Temperature distribution in the adiabatic 'five-element machine'. The contact with the surroundings is effectuated by the cooler and the freezer. The mean temperature in the compression space $T_{C,a}$ is higher than the cooler temperature T_C as heat must be transported from the cylinder to the cooler; by the same argument the mean temperature in the expansion space $T_{E,a}$ is lower than that of the freezer T_E.*

heat exchangers, which causes an increase in shaft power. Strictly speaking, the expression "adiabatic loss" is therefore misleading and should be replaced by "transport loss". It is adhered to though, because ultimately the adiabatic behavior is the fundamental cause that has necessitated the introduction of separate heat exchangers. Because the value of the temperature ratio τ is larger in the adiabatic than in the isothermal case, one would expect also a decrease in refrigerating capacity; this influence is very small, however, as the larger value of τ is nearly compensated by a decrease of the quantity B, the mean relative volume of the working circuit, which quantity has another form in the adiabatic case.

In the above discussion the use of the expressions "adiabatic compression" or "expansion has been intentionally avoided. The reason is that the gas is compressed and expanded everywhere in the working space; its behavior during these processes, however, depends largely on the condition of heat transfer prevailing in the various sections of the volume. While in the cylinders the gas temperatures change nearly adiabatically, this is not the case in the connecting circuit;

in the regenerator, for example, the heat transfer is so high, that the behavior is practically isothermal. Thus, while the actual Stirling process is certainly not isothermal, it would be incorrect to call it adiabatic. This situation thus furnishes still another example for the previously made remark that thermodynamic diagrams are of little value for this process.

The decrease of the refrigerating capacity is mainly due to two effects, the flow loss and the insulation loss. Again no comment is made on the flow loss. Also the insulation (and conduction) loss proper needs no discussion. There exists another loss, however, that acts as an insulation loss as well which has the utmost importance for the quality of the process. This loss, caused by the non-ideal behavior of the renerator, will now be discussed.

It is obvious that in the Stirling process ideal regeneration is possible in principle as the same amount of gas passes the regenerator in both directions with the same temperature difference, so that the amount of heat rejected and absorbed by the gas for both directions of flow is the same, since the specific heat is independent of pressure (which is practically the case for nearly perfect gases). The following argument shows the extreme importance of a small departure from ideality of the regenerator, caused by non-ideal heat transfer. In the regenerator a quantity of heat Q_r must be absorbed and rejected in each cycle. Owing to imperfections, this amount is reduced to $\eta_r Q_r$, where η_r is the efficiency of the regenerator. This means that only part of the available heat is transferred to the regenerator. The rest, that is $(1-\eta_r)Q_r$, is carried along with the gas through the regenerator, so that the gas arrives too hot in the expansion space. This remainder, ΔQ_r, thus constitutes the regeneration loss. As this loss must be made up from the cold produced Q_E, it must be compared with this quantity. Calculations show that

$$\frac{\Delta Q_r}{Q_E} = C_r(1-\eta_r)\frac{T_C - T_E}{T_E} \qquad \ldots(7)$$

The constant C_r depends mainly on the compression ratio; its value is approximately 10. For example, take T_C = 300°K, T_E = 75°K, and the regeneration loss $1-\eta_r$ = 1%; then

$$\frac{\Delta Q_r}{Q_E} = 10 \times 1 \times 3 = 30 \text{ per cent}$$

Thus each percent of regeneration loss involves a loss of 30 per cent in refrigerating power; below 30°K this figure even increases to greater than 90 percent, meaning that at such temperatures the entire cold production is consumed by the regenerator. It is thus no exaggeration to call the regenerator the heart of the machine.

The regenerator used in actual gas refrigerating machines consists of a mass of fine metal wire, forming a light felt-like substance. With this type of material, efficiencies of 99 per cent and higher can be obtained; the thermal conductivity of the material is very low.

This section will be ended with a short discussion on the problem of how to minimize the total sum of the losses; this problem is very involved indeed, so that only a very broad outline can be given. As an example, the regenerator will be considered. The losses of the regenerator make themselves felt in three different ways, viz., the regeneration loss (reducing the cooling power), the flow loss, and the dead space (this is not a real loss as the shaft power is also decreased; it exerts its influence through the other losses, as these increase relatively). The regeneration loss can be reduced by making a longer regnerator; this, however, increases the flow loss and the dead space. One may also give the regenerator a larger cross-section; this reduces the flow loss and somewhat the regeneration loss, but increases the dead space. Thus it is possible to find optimum dimensions for the regenerator. The same hold for the freezer and the cooler, where the regeneration loss is replaced by the loss due to insufficient heat transfer. Moreover, the losses are governed by the choice of the values of w and ϕ. Finally, the

outside of the cooler and the freezer must be made optimal. Thus a very large number of design parameters are to be fixed such that the total result gives an optimum condition; it will be obvious that a lot of experience is needed to find the right solution quickly. This is compensated by the fact that once a system of calculation is worked out, it holds for any size of machine.